Minimal degree of Genocchi-Peano function of *n* variables

Krzysztof Chris Ciesielski

Department of Mathematics, West Virginia University
MIPG, Department of Radiology, University of Pennsylvania

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Outline

1. (Pre)Historical background
2. From calculus examples to number theoretical problems
3. Good estimates for $D(n)$
4. Remaining open problems (on $D_b(n)$ and $D(n)$)

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Cauchy's mistake?

$f\colon \mathbb{R}^2 \to \mathbb{R}$ is separately continuous iff
all maps $t \mapsto f(t, y)$ and $t \mapsto f(x, t)$ are continuous.

A theorem in 1821 textbook *Cours d'analyse* by Cauchy:

A separately cont function of real variables is continuous.

A counterexample, 1884 calculus text by Genocchi and Peano,
included also in the calculus text we currently use:

$$g(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } \langle x,y\rangle \neq \langle 0,0\rangle \\ 0 & \text{for } \langle x,y\rangle = \langle 0,0\rangle \end{cases}$$

is separately continuous but discontinuous on $y = x$.

Did Cauchy make mistake?

Cauchy was right!!!

YES! No contradiction, since Cauchy's text was written for the set $\mathcal{R}$ of real numbers containing infinitesimals, rather than nowadays standard set $\mathbb{R}$ of reals.

Not surprisingly, since $\mathbb{R}$ firmly replaced $\mathcal{R}$ in analysis (in the mid 19th century) the interrelation between continuity and (generalized) separate continuity was intensely studied.

In particular, the subject was studied, among others, by E. Heine, H. Lebesgue, G. Peano, R. Baire, W. Sierpiński, N. Luzin, E. Marczewski, and A. Rosenthal.

Another Genocchi-Peano example

$$f(x,y)=\begin{cases}\frac{xy^2}{x^2+y^4} & \text{when } (x,y)\neq(0,0)\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

This function f is discontinuous (along $x=y^2$), but its restriction to any line (i.e., a hyperplane in $\mathbb{R}^2$) is continuous.

For more on this history, see

K.C. Ciesielski and D. Miller, *A continuous tale on continuous and separately continuous functions*, Real Anal. Exchange **41**(1) (2016), 19-54,

see http://www.math.wvu.edu/˜kcies/publications.html

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Generalizations of $\frac{xy^2}{x^2+y^4}$ to more variables

$$g(x) = \begin{cases} \frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}} & \text{when } x \neq (0,0,\ldots,0), \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

is a *Genocchi-Peano example, GPE*, if g is discontinuous but has continuous restriction to any hyperplane in $\mathbb{R}^n$.

Paper (see http://www.math.wvu.edu/˜kcies/publications.html)

K.C. Ciesielski and D. Miller, *On a Genocchi-Peano example*, College Math. J., to appear

contains the following characterization of GPEs:

GPEs characterization theorem

Theorem (KC&DM)

Let $g(x) = \frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}}$ *and* $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$ *be even.*

(i) *g is discontinuous iff* $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} \leq 1$.

(ii) *$g \restriction H$ is continuous for every hyperplane H iff*

$$\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} > 1 \text{ for every } k \in \{2, \ldots, n\}. \tag{3}$$

So, g is a GPE iff $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} \leq 1$ *and (3) holds. Moreover,*

(iii) *g is a* **bounded** *GPE iff* $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = 1$ *and all β_is are distinct.*

Proof uses only elementary calculus tools.
Good exercise for math 451 or honors section of math 251.

Corollary: simple GPEs

g is a bounded GPE iff $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = 1$ and all β_is are distinct

immediately implies that the following maps are bounded GPEs:

$$\frac{x_1 x_2 \cdots x_{n-1} x_n^2}{x_1^2 + x_2^4 + \cdots + x_{n-1}^{2n-1} + x_n^{2^n}}$$

$$\frac{x_1^2 \cdots x_i^{2i} \cdots x_n^{2n}}{x_1^{2n} + \cdots + x_i^{2in} + \cdots + x_n^{2n^2}}$$

Search for "minimal" GPEs of n-variables

For GPE $g(x) = \frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}}$ its degree is defined $D(g) = \beta_n$.

Define

$$D(n) = \min\{D(g) \colon g \text{ is a GPE of } n \text{ variables}\}$$

$$D_b(n) = \min\{D(g) \colon g \text{ is a bounded GPE of } n \text{ variables}\}$$

General problem: Find as much as possible on $D(n)$ & $D_b(n)$.

By KC&DM theorem, this is a number theoretical problem.

Easy bonds: $2n \leq D(n) \leq D_b(n) \leq \min\{2^n, 2n^2\}$.

$D(n)$ is discussed below. Almost nothing else is known about $D_b(n)$, except that $D_b(n) \leq n(n+1)$.

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The values of $D(n)$s

Let $k_n = \min\left\{k \in \omega\colon \sum_{i=1}^{n} \frac{1}{k+i} \leq 2\right\}$.

Theorem (KC, proved last month)

For every $n = 2, 3, 4, \ldots$ *we have*

$$k_n \in \left\{ \left\lfloor \frac{1}{e^2-1} n \right\rfloor, \left\lceil \frac{1}{e^2-1} n \right\rceil \right\} \tag{4}$$

and

$$D(n) \in \{2(k_n+n), 2(k_n+n)+2\}. \tag{5}$$

In particular, for some $i_n \in \{0, 2, 4\}$,

$$D(n) = 2\left\lfloor \frac{e^2}{e^2-1} n \right\rfloor + i_n \in \left(\frac{2e^2}{e^2-1} n - 2, \frac{2e^2}{e^2-1} n + 4 \right) \subset (2.31n - 2, 2.32n + 4).$$

Lemmas needed in the proof of the new theorem

Proposition (A)

Let $g(x) = \frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}}$ and numbers $\beta_1 < \beta_2 < \cdots < \beta_n$ be even. If $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} \leq 1 < \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} + \frac{2}{\beta_n(\beta_n - 2)}$, then g is a GPE.

Proof.

g clearly satisfies (i) of KC&DM theorem. It satisfies (ii) since
$\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} + \alpha_k \frac{\beta_k - \beta_{k-1}}{\beta_k \beta_{k-1}} \geq$
$\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} + \frac{2}{\beta_n(\beta_n - 2)} > 1$. □

The main proposition

Proposition (B)

Let $k, n < \omega$ and $\beta_i = 2(k+i)$. If $\sum_{i=1}^{n} \frac{1}{\beta_i} + \frac{4}{\beta_n} \leq 1$ & $n \geq k+2$, then there exist α_is such that $\frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}}$ is a GPE.

PROOF. Need to find α_i's satisfying assumptions of Prop A.

Step 1: Let $\alpha_i = 1$ for all $i < n$ and α_n be the largest s.t. $S_0 = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = \sum_{i=1}^{n-1} \frac{1}{\beta_i} + \frac{\alpha_n}{\beta_n} \leq 1$. Note that $\alpha_n \geq 5$.

If $S_0 + \frac{1}{\beta_{n-1}} - \frac{1}{\beta_n} > 1$, then, by Proposition A, we are done as $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} + \frac{2}{\beta_n(\beta_n - 2)} = S_0 + \frac{1}{\beta_{n-1}} - \frac{1}{\beta_n} > 1$.

So, assume that $S_0 + \frac{1}{\beta_{n-1}} - \frac{1}{\beta_n} \leq 1$.

Continuation of the proof of Proposition B

Step 2: Pick the smallest $j \le n-1$ with $S_0 + \frac{1}{\beta_j} - \frac{1}{\beta_n} \le 1$.

By maximality of α_n, $S_0 + \frac{1}{\beta_j} - \frac{1}{\beta_n} \le 1 < S_0 + \frac{1}{\beta_n}$.

So, $\beta_n/2 < \beta_j$. In particular, $j > 1$. (Otherwise $k + n = \beta_n/2 < \beta_1 = 2(k+1)$, contradicting $n \ge k+2$.)

Modify α_i's by putting $\alpha_j = 2$ and decreasing α_n by 1. Then,

$$S_1 = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = S_0 + \frac{1}{\beta_j} - \frac{1}{\beta_n} \le 1 < S_0 + \frac{1}{\beta_{j-1}} - \frac{1}{\beta_n} = S_1 + \frac{1}{\beta_{j-1}} - \frac{1}{\beta_j}.$$

$\beta_n/2 < \beta_j$ implies $\beta_{n-1}/2 = (\beta_n/2) - 1 \le \beta_j - 2 = \beta_{j-1}$. So

$$1 < S_1 + \frac{1}{\beta_{j-1}} - \frac{1}{\beta_j} = S_1 + \frac{2}{\beta_{j-1}\beta_j} < S_1 + \frac{2}{\frac{\beta_{n-1}}{2}\frac{\beta_n}{2}} = S_1 + 4\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right)$$

Completion of the proof of Proposition B

Step 3: As $S_1 \leq 1 < S_1 + 4\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right)$ there is $m \leq 3$ s.t.

$$S_1 + m\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right) \leq 1 < S_1 + (m+1)\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right)$$

Modify α_i's by decreasing α_n by m (we will still have $\alpha_n \geq 1$) and increasing the previous value of α_{n-1} by m.

These new α_i's satisfy assumptions of Proposition A, as
$\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = S_1 + m\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right) \leq 1 <$
$S_1 + (m+1)\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right) = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} + \frac{2}{\beta_n(\beta_n - 2)}$, as required.

One more lemma

Recall that $k_n = \min\left\{k \in \omega\colon \sum_{i=1}^{n} \frac{1}{k+i} \leq 2\right\}$.

Lemma

(4) from Theorem holds, as $k_n \in \left(\frac{1}{e^2-1}n - 1, \frac{1}{e^2-1}n + 1\right)$*.*
Moreover, $\lim_{n\to\infty} \frac{1}{e^2-1} n/k_n = 1$*.*

Sketch of proof.

$$\ln\left(1 + \frac{n}{k+1}\right) = \int_{k+1}^{k+1+n} \frac{1}{x}\, dx < \sum_{i=1}^{n} \frac{1}{k+i} < \int_{k}^{k+n} \frac{1}{x}\, dx = \ln\left(1 + \frac{n}{k}\right)$$

$\sum_{i=1}^{n} \frac{1}{k+i} \leq 2$ is ensured when $\ln\left(1 + \frac{n}{k}\right) \leq 2$, i.e., $k \geq \frac{1}{e^2-1}n$.
So, $k_n < \frac{1}{e^2-1}n + 1$.

$\sum_{i=1}^{n} \frac{1}{k+i} \leq 2$ is false when $2 \leq \ln\left(1 + \frac{n}{k+1}\right)$, i.e.,
$k \leq \frac{1}{e^2-1}n - 1$. Hence, $k_n \geq \frac{1}{e^2-1}n - 1$.

(The case when $k = 0$ needs to be considered separately.) □

Proof of the theorem

Theorem (reminder, main parts)

$$k_n \in \left\{ \left\lfloor \frac{1}{e^2-1} n \right\rfloor, \left\lceil \frac{1}{e^2-1} n \right\rceil \right\} \quad (6)$$

$$D(n) \in \{2(k_n+n), 2(k_n+n)+2\} \quad (7)$$

PROOF. (6) was proved in the lemma.

$D(n) \geq 2(k_n+n)$: Let $\frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1}+x_2^{\beta_2}+\cdots+x_n^{\beta_n}}$ be a GPE with $D(n) = D(g) = \beta_n = 2m$. By Theorem KC&DM, $\beta_1 < \cdots < \beta_n$ are even and $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} \leq 1$. Hence $\beta_{n-i} \leq 2(m-i)$ and

$$1 \geq \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} \geq \sum_{i=1}^{n} \frac{1}{\beta_i} \geq \sum_{i=0}^{n-1} \frac{1}{2(m-i)} = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{(m-n)+i}.$$

So, $\sum_{i=1}^n \frac{1}{(m-n)+i} \leq 2$, $k_n \leq m-n$, and $D(n) = 2m \geq 2(k_n+n)$.

Proof of $D(n) \leq 2(k_n + n) + 2$

First this is proved for $n \notin E = \{2, 3, 4, 5, 6, 7, 10, 11\}$.

Note that $k = k_n + 1$ and $n \notin E$ satisfy assumptions of Prop. B.

$n \geq k + 2$: For $k = k_n + 1$ it becomes $n - k_n \geq 3$. But this holds for any $n \geq 8$, since $\frac{1}{e^2-1} < 0.2$ and, by the lemma, $k_n < \frac{1}{e^2-1} n + 1$, so that
$n - k_n > n - \left(\frac{1}{e^2-1} n + 1\right) > 0.8n - 1 \geq 0.8 \cdot 8 - 1 > 3$.

$\sum_{i=1}^n \frac{1}{\beta_i} + \frac{4}{\beta_n} \leq 1$, **where** $\beta_i = 2(k+i)$: $\sum_{i=1}^n \frac{1}{k_n+i} \leq 2$. So
$\sum_{i=1}^n \frac{1}{k+i} = \sum_{i=1}^n \frac{1}{k_n+i} + \frac{1}{k_n+1+n} - \frac{1}{k_n+1} \leq 2 - \left(\frac{1}{k_n+1} - \frac{1}{k_n+1+n}\right)$.
By this and $\frac{1}{k_n+1} - \frac{1}{k_n+1+n} = \left(\frac{n}{k_n+1}\right)\frac{1}{k_n+1+n} = \left(\frac{n}{k_n+1}\right)\frac{2}{\beta_n}$ we see that $\sum_{i=1}^n \frac{1}{\beta_i} = \frac{1}{2}\sum_{i=1}^n \frac{1}{k+i} \leq 1 - \left(\frac{n}{k_n+1}\right)\frac{1}{\beta_n}$, that is,
$\sum_{i=1}^n \frac{1}{\beta_i} + \left(\frac{n}{k_n+1}\right)\frac{1}{\beta_n} \leq 1$.

Need $\sum_{i=1}^{n} \frac{1}{\beta_i} + \frac{4}{\beta_n} \leq 1$; have $\sum_{i=1}^{n} \frac{1}{\beta_i} + \left(\frac{n}{k_n+1}\right) \frac{1}{\beta_n} \leq 1$

It is enough to show that

$$\frac{n}{k_n + 1} \geq 4 \quad \text{for any } n \notin E. \tag{8}$$

To see (8), note that $\frac{n}{\frac{1}{e^2-1}n+2} \geq 4$ is equivalent to $n \geq \frac{8}{1-4\frac{1}{e^2-1}}$ which holds for $n \geq 22$, since $22 > 21.4 > \frac{8}{1-4\frac{1}{e^2-1}}$. So, (8) holds for any $n \geq 22$ as, using $k_n < \frac{1}{e^2-1}n + 1$, we have $\frac{n}{k_n+1} > \frac{n}{\left(\frac{1}{e^2-1}n+1\right)+1} = \frac{n}{\frac{1}{e^2-1}n+2} \geq 4$.

n	8	9	10	11	12	13	14	15	16	17	18	19	20	21
k_n	1	1	2	2	2	2	2	2	2	3	3	3	3	3

For the remaining values of $n \notin E$, (8) is justified by the table.

End of proof: for $n \in \{2,3,4,5,6,7,10,11\}$ see table

n	k_n	a GPE g of n variables	D(g)	$2(k_n+n)+2$
2	0	$\frac{x_1^1 x_2^2}{x_1^2+x_2^4}$	4	6
3	0	$\frac{x_1^1 x_2^3 x_3^2}{x_1^4+x_2^6+x_3^8}$	8	8
4	1	$\frac{x_1^2 x_2^1 x_3^1 x_4^2}{x_1^4+x_2^6+x_3^8+x_4^{10}}$	10	12
5	1	$\frac{x_1^1 x_2^1 x_3^1 x_4^2 x_5^3}{x_1^4+x_2^6+x_3^8+x_4^{10}+x_5^{12}}$	12	14
6	1	$\frac{x_1^1 x_2^1 x_3^2 x_4^1 x_5^1 x_6^2}{x_1^4+x_2^6+x_3^8+x_4^{10}+x_5^{12}+x_6^{14}}$	14	16
7	1	$\frac{x_1^1 x_2^1 x_3^1 x_4^1 x_5^1 x_6^2 x_7^2}{x_1^4+x_2^6+x_3^8+x_4^{10}+x_5^{12}+x_6^{14}+x_7^{16}}$	16	18
10	2	$\frac{x_1^1 x_2^1 x_3^1 x_4^1 x_5^2 x_6^1 x_7^1 x_8^1 x_9^1 x_{10}^4}{x_1^6+x_2^8+x_3^{10}+x_4^{12}+x_5^{14}+x_6^{16}+x_7^{18}+x_8^{20}+x_9^{22}+x_{10}^{24}}$	24	26
11	2	$\frac{x_1^1 x_2^1 x_3^1 x_4^1 x_5^1 x_6^1 x_7^1 x_8^1 x_9^1 x_{10}^2 x_{11}^4}{x_1^6+x_2^8+x_3^{10}+x_4^{12}+x_5^{14}+x_6^{16}+x_7^{18}+x_8^{20}+x_9^{22}+x_{10}^{24}+x_{11}^{26}}$	26	28

Table: GPEs of $n \in E$ variables with degrees $\leq 2(k_n+n)+2$.

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Open problems on $D(n)$

Problem (1)

What can be said about the sets
$D_i = \left\{ n \geq 2\colon D(n) = 2\left\lfloor \frac{e^2}{e^2-1} n \right\rfloor + 2i \right\}$, *where* $i \in \{0, 1, 2\}$*?*
Are they all infinite?
Notice that the structure of sets D_i *is related to the structure of sets* $K_i = \left\{ n \geq 2\colon k_n = \left\lfloor \frac{1}{e^2-1} n \right\rfloor + i \right\}$, *where* $i \in \{0, 1\}$, *since* $K_0 \subset D_0 \cup D_1$ *and* $K_1 \subset D_1 \cup D_2$*.*

Clearly, $D(n) \leq 2(k_n + n + 1) \leq 2(k_{n+1} + (n+1)) \leq D(n+1)$.

Problem (2)

How big is the set $E = \{n \geq 2\colon D(n) = D(n+1)\}$*? Is it infinite?*

Notice that $E \neq \emptyset$ since $14 \in E$: $D(14) = D(15) = 34$.

Open problems on $D_b(n)$

The values of $D_b(n)$s are considerably harder to estimate.
$D(n) \leq D_n(n) \leq \min\{2^n, n(n+1)\}$
are essentially the best estimates we have.

Problem

Is it possible to express the numbers $D_b(n)$ in algebraic terms in terms of n? If not, is it possible at least of find the upper and lower bounds of these numbers with the same order $O(n^\delta)$ of magnitude?

Problem

What can be shown about the set $Z = \{n \geq 2\colon D_b(n) = D(n)\}$? In particular, is it finite? infinite?

Notice, that $2, 3 \in Z$ but $4, 5, 6, 7, 8 \notin Z$.
The examples for $D(n)$ from Prop (B) cannot work for $D_b(n)$!

That is all!

Thank you for your attention!

Extras: how to prove Theorem KC&DM

Theorem (KC&DM)

Let $g(x) = \frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}}$ *and* $\beta_1 \le \beta_2 \le \cdots \le \beta_n$ *be even.*

(i) *g is discontinuous iff* $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} \le 1$.

(ii) *$g \restriction H$ is continuous for every hyperplane H iff*

$$\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} > 1 \text{ for every } k \in \{2, \ldots, n\}. \qquad (9)$$

So, g is a GPE iff $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} \le 1$ *and (3) holds. Moreover,*

(iii) *g is a bounded GPE iff* $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = 1$ *and all β_is are distinct.*

Extras: how to prove Theorem KC&DM

Part (i) of the theorem follows from the fact that, for $\gamma = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i}$ and $d = x_1^{\beta_1} + \cdots + x_n^{\beta_n}$, we have $|g(x_1, \ldots, x_n)| \leq d^{\gamma-1}$ and $g\left(t^{1/\beta_1}, \ldots, t^{1/\beta_n}\right) = \frac{t^{\gamma-1}}{n}$. To see the necessity of (9) it is enough to notice that, for $\delta_k = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}}$ and $f_i(t)$ defined as t^{1/β_i} for $i \neq k$ and as $t^{1/\beta_{k-1}}$ for $i = k$, we have the equality $g(f_1(t), \ldots, f_n(t)) = \frac{1}{(n-1)+t^{(\beta_k/\beta_{k-1})-1}} t^{\delta_k - 1}$.

The condition (3) is sufficient since, for every hyperplane given by an equation $x_k = \sum_{i=1}^{k-1} a_i x_i$, we have $|g(x_1, \ldots, x_n)| \leq A^{\alpha_k} d^{\delta_k - 1}$, where $A = \sum_{i=1}^{k-1} |a_i|$. Finally, the boundedness claim is justified by $g(x_1, \ldots, x_n) = \frac{1}{d^{1-\gamma}} \prod_{i=1}^{n} \frac{(x_i)^{\alpha_i}}{d^{\alpha_i/\beta_i}}$.